

The Spectral Action Principle

Ali H. Chamseddine^{1,2} and Alain Connes²

1. Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland
2. I.H.E.S., F-91440 Bures-sur-Yvette, France

Abstract.

We propose a new action principle to be associated with a noncommutative space $(\mathcal{A}, \mathcal{H}, D)$. The universal formula for the spectral action is $(\psi, D\psi) + \text{Trace}(\chi(D/\Lambda))$ where ψ is a spinor on the Hilbert space, Λ is a scale and χ a positive function. When this principle is applied to the noncommutative space defined by the spectrum of the standard model one obtains the standard model action coupled to Einstein plus Weyl gravity. There are relations between the gauge coupling constants identical to those of $SU(5)$ as well as the Higgs self-coupling, to be taken at a fixed high energy scale.

1. Introduction.

The basic data of Riemannian geometry consists in a *manifold* M whose points $x \in M$ are locally labelled by finitely many coordinates $x^\mu \in \mathbb{R}$, and in the infinitesimal *line element*, ds ,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu . \quad (1.1)$$

The laws of physics at reasonably low energies are well encoded by the action functional,

$$I = I_E + I_{SM} \quad (1.2)$$

where $I_E = \frac{1}{16\pi G} \int R \sqrt{g} d^4x$ is the Einstein action, which depends only upon the 4-geometry (we shall work throughout in the Euclidean, i.e. imaginary time formalism) and where I_{SM} is the standard model action, $I_{SM} = \int \mathcal{L}_{SM}$, $\mathcal{L}_{SM} = \mathcal{L}_G + \mathcal{L}_{GH} + \mathcal{L}_H + \mathcal{L}_{Gf} + \mathcal{L}_{Hf}$. The action functional I_{SM} involves, besides the 4-geometry, several additional fields: bosons G of spin 1 such as γ , W^\pm and Z , and the eight gluons, bosons of spin 0 such as the Higgs field H and fermions f of spin 1/2, the quarks and leptons.

These additional fields have *a priori* a very different status than the geometry (M, g) and the gauge invariance group which governs their interaction is *a priori* very different from the diffeomorphism group which governs the invariance of the Einstein action. In fact the natural group of invariance of the functional (1.2) is the semidirect product,

$$G = \mathcal{U} \rtimes \text{Diff}(M) \quad (1.3)$$

of the group of local gauge transformations, $\mathcal{U} = C^\infty(M, U(1) \times SU(2) \times SU(3))$ by the natural action of $\text{Diff}(M)$.

The basic data of noncommutative geometry consists of an involutive *algebra* \mathcal{A} of operators in Hilbert space \mathcal{H} and of a selfadjoint unbounded operator D in \mathcal{H} [1-6].

The inverse D^{-1} of D plays the role of the infinitesimal unit of length ds of ordinary geometry.

To a Riemannian compact spin manifold corresponds the *spectral triple* given by the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on M , the Hilbert space $\mathcal{H} = L^2(M, S)$ of L^2 -spinors and the Dirac operator D of the Levi-Civita Spin connection. The line element ds is by construction the propagator of fermions,

$$ds = \not{x} \not{x} . \quad (1.4)$$

No information is lost in trading the original Riemannian manifold M for the corresponding spectral triple $(\mathcal{A}, \mathcal{H}, D)$. The points of M are recovered as the characters of the involutive algebra \mathcal{A} , i.e. as the homomorphisms $\rho : \mathcal{A} \rightarrow \mathbb{C}$ (linear maps such that $\rho(ab) = \rho(a)\rho(b) \quad \forall a, b \in \mathcal{A}$). The geodesic distance between points is recovered by

$$d(x, y) = \text{Sup} \{ |a(x) - a(y)| ; a \in \mathcal{A} , \|[D, a]\| \leq 1 \} . \quad (1.5)$$

More importantly one can characterize the spectral triples $(\mathcal{A}, \mathcal{H}, D)$ which come from the above spinorial construction by very simple axioms ([4]) which involve the dimension n of M . The parity of n implies a $\mathbb{Z}/2$ grading γ of the Hilbert space \mathcal{H} such that,

$$\gamma = \gamma^* , \quad \gamma^2 = 1 , \quad \gamma a = a \gamma \quad \forall a \in \mathcal{A} , \quad \gamma D = -D \gamma . \quad (1.6)$$

Moreover one keeps track of the *real structure* on \mathcal{H} as an antilinear isometry J in \mathcal{H} satisfying simple relations

$$J^2 = \varepsilon , \quad J D = \varepsilon' D J , \quad J \gamma = \varepsilon'' \gamma J ; \quad \varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\} \quad (1.7)$$

where the value of $\varepsilon, \varepsilon', \varepsilon''$ is determined by n modulo 8. One first virtue of these axioms is to allow for a shift of point of view, similar to Fourier transform, in which the usual emphasis on the points $x \in M$ of a geometric space is now replaced by the spectrum $\Sigma \subset \mathbb{R}$ of the operator D . Indeed, if one forgets about the algebra \mathcal{A} in the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ but retains only the operators D , γ and J acting in \mathcal{H} one can (using (1.7)) characterize this data by the spectrum Σ of D which is a discrete subset with multiplicity of \mathbb{R} . In the even case $\Sigma = -\Sigma$. The existence of Riemannian manifolds which are isospectral (i.e. have the same Σ) but not isometric shows that the following hypothesis is stronger than the usual diffeomorphism invariance of the action of general relativity,

$$\text{“The physical action only depends upon } \Sigma \text{.”} \quad (1.8)$$

In order to apply this principle to the action (1.2) we need to exploit a second virtue of the axioms (cf. [4]) which is that they do not require the commutativity of the algebra \mathcal{A} . Instead one only needs the much weaker form,

$$ab^0 = b^0 a \quad \forall a, b \in \mathcal{A} \quad \text{with} \quad b^0 = J b^* J^{-1} . \quad (1.9)$$

In the usual Riemannian case the group $\text{Diff}(M)$ of diffeomorphisms of M is canonically isomorphic to the group $\text{Aut}(\mathcal{A})$ of automorphisms of the algebra $\mathcal{A} = C^\infty(M)$. To each $\varphi \in \text{Diff}(M)$ one associates the algebra preserving map $\alpha_\varphi : \mathcal{A} \rightarrow \mathcal{A}$ given by

$$\alpha_\varphi(f) = f \circ \varphi^{-1} \quad \forall f \in C^\infty(M) = \mathcal{A}. \quad (1.10)$$

In general the group $\text{Aut}(\mathcal{A})$ of automorphisms of the involutive algebra \mathcal{A} plays the role of the diffeomorphisms of the noncommutative (or spectral for short) geometry $(\mathcal{A}, \mathcal{H}, D)$. The first interesting new feature of the general case is that the group $\text{Aut}(\mathcal{A})$ has a natural normal subgroup,

$$\text{Int}(\mathcal{A}) \subset \text{Aut}(\mathcal{A}) \quad (1.11)$$

where an automorphism α is *inner* iff there exists a unitary operator $u \in \mathcal{A}$, ($uu^* = u^*u = 1$) such that,

$$\alpha(a) = uau^* \quad \forall a \in \mathcal{A}. \quad (1.12)$$

The corresponding exact sequence of groups,

$$1 \rightarrow \text{Int}(\mathcal{A}) \rightarrow \text{Aut}(\mathcal{A}) \rightarrow \text{Out}(\mathcal{A}) \rightarrow 1 \quad (1.13)$$

looks very similar to the exact sequence

$$1 \rightarrow \mathcal{U} \rightarrow G \rightarrow \text{Diff}(M) \rightarrow 1 \quad (1.14)$$

which describes the structure of the symmetry group G of the action functional (1.2).

Comparing (1.13) and (1.14) and taking into account the action of inner automorphisms of \mathcal{A} in \mathcal{H} given by

$$\xi \rightarrow u(u^*)^0 \xi = u \xi u^* \quad (1.15)$$

one determines the algebra \mathcal{A} such that $\widetilde{\text{Aut}}(\mathcal{A}) = G$ (where $\widetilde{\text{Aut}}$ takes into account the action of automorphisms in the Hilbert space \mathcal{H}). The answer is

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F \quad (1.16)$$

where the algebra \mathcal{A}_F is *finite dimensional*,

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \quad (1.17)$$

where $\mathbb{H} \subset M_2(\mathbb{C})$ is the algebra of quaternions,

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} ; \alpha, \beta \in \mathbb{C} \right\}. \quad (1.18)$$

Giving the algebra \mathcal{A} does not suffice to determine the spectral geometry, one still needs the action of \mathcal{A} in \mathcal{H} and the operator D . Since \mathcal{A} is a tensor product (16) which geometrically corresponds to a product space, an instance of spectral geometry for \mathcal{A} is given by the product rule,

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F, \quad D = \not{D}_M \otimes 1 + \gamma_5 \otimes D_F \quad (1.19)$$

where (\mathcal{H}_F, D_F) is a spectral geometry on \mathcal{A}_F , while both $L^2(M, S)$ and the Dirac operator \not{D}_M on M are as above.

Since \mathcal{A}_F is finite dimensional the dimension of the corresponding space is 0 so that \mathcal{H}_F must be finite dimensional. The list of elementary fermions provides a natural candidate for \mathcal{H}_F . One lets \mathcal{H}_F be the Hilbert space with basis labelled by elementary leptons and quarks. Thus for the first generation of leptons we get $e_L, e_R, \nu_L, \bar{e}_L, \bar{e}_R, \bar{\nu}_L$ for instance, as the corresponding basis. The $\mathbb{Z}/2$ grading γ_F is given by $+1$ for left handed particles and -1 for right handed ones. For quarks one has an additional color index, y, r, b . The involution J is just such that $Jf = \bar{f}$ for any f in the basis. One has $J^2 = 1$, $J\gamma = \gamma J$ as dictated by the dimension $n = 0$. Moreover the algebra \mathcal{A}_F has a natural representation in \mathcal{H}_F and:

$$ab^0 = b^0 a \quad \forall a, b \in \mathcal{A}_F, \quad b^0 = Jb^* J^{-1}. \quad (1.20)$$

Finally there is a natural matrix acting in the finite dimensional Hilbert space \mathcal{H}_F . It is

$$D_F = \begin{bmatrix} Y & 0 \\ 0 & \bar{Y} \end{bmatrix}, \quad (1.21)$$

where Y is the Yukawa coupling matrix.

The special features of Y show that the algebraic rule

$$[[D, a], b^0] = 0 \quad \forall a, b \in \mathcal{A} \quad (1.22)$$

which is one of the essential axioms, holds for the spectral geometry $(\mathcal{A}_F, \mathcal{H}_F, D_F) = F$. Of course this 0-dimensional geometry is encoding the knowledge of the fermions of the standard model and it is a basic question to understand and characterize it abstractly, but let us postpone this problem and proceed with the product geometry $M \times F$.

The next important new feature of the noncommutative case is the following. We saw that the group $\text{Aut}(\mathcal{A})$ of diffeomorphisms falls in equivalence classes under the normal subgroup $\text{Int}(\mathcal{A})$ of inner automorphisms. In the same way the space of metrics has a natural foliation into equivalence classes. The *internal fluctuations* of a given metric are given by the formula,

$$D = D_0 + A + JAJ^{-1} \ , \ A = \Sigma a_i[D_0, b_i] \ , \ a_i, b_i \in \mathcal{A} \text{ and } A = A^* . \quad (1.23)$$

Thus starting from $(\mathcal{A}, \mathcal{H}, D_0)$ with obvious notations, one leaves the representation of \mathcal{A} in \mathcal{H} untouched and just perturbs the operator D_0 by (1.23) where A is an arbitrary self-adjoint operator in \mathcal{H} of the form $A = \Sigma a_i[D_0, b_i]$; $a_i, b_i \in \mathcal{A}$. One checks that this does not alter the axioms (check (1.22) for instance). These fluctuations are trivial: $D = D_0$ in the usual Riemannian case in the same way as the group of inner automorphisms $\text{Int}(\mathcal{A}) = \{1\}$ is trivial for $\mathcal{A} = C^\infty(M)$.

In general the natural action of $\widetilde{\text{Int}}(\mathcal{A})$ on the space of metrics restricts to the above equivalence classes and is simply given by (for the automorphism associated to $u \in \mathcal{A}$, $uu^* = u^*u = 1$),

$$\xi \in \mathcal{H} \rightarrow u\xi u^* = uu^{*0} \xi \ , \ A \rightarrow u[D, u^*] + uAu^* . \quad (1.24)$$

When one computes the internal fluctuations of the above product geometry $M \times F$ one finds ([6]) that they are parametrized exactly by the bosons γ, W^\pm, Z , the eight gluons and the Higgs fields H of the standard model. The equality

$$\int_M (\mathcal{L}_{Gf} + \mathcal{L}_{Hf}) \sqrt{g} d^4x = \langle \psi, D\psi \rangle \quad (1.25)$$

gives the contribution to (1.2) of the last two term of the *SM* Lagrangian in terms of the operator D alone.

The operator D encodes the metric of our “discrete Kaluza Klein” geometry $M \times F$ but this metric is no longer the product metric as it was for D_0 . In fact the initial scale given by D_F completely disappears when one considers the arbitrary internal fluctuations of $D_0 = \not{D}_M \otimes 1 + \gamma_5 \otimes D_F$. What remains is to understand in a purely

gravitational manner the 4 remaining terms of the action (1.2). This is where we apply the basic principle (1.8).

We shall check in this paper that for any smooth cutoff function χ , $\chi(\lambda) = 1$ for $|\lambda| \leq 1$, one has

$$\text{Trace } \chi \left(\frac{D}{\Lambda} \right) = I_E + I_G + I_{GH} + I_H + I_C + o(\Lambda^{-\infty}) \quad (1.26)$$

where I_C is a sum of a cosmological term, a term of Weyl gravity and a term in $\int R H^2 \sqrt{g} d^4x$. The computation in itself is not new, and goes back to the work of DeWitt [7]. Similar computations also occur in the theory of induced gravity [8]. It is clear that the left hand side of (1.26) only depends upon the spectrum Σ of the operator D , and following our principle (1.8) this allows to take it as the natural candidate for the bare action at the cutoff scale Λ .

In our framework there is a natural way to cutoff the geometry at a given energy scale Λ , which has been developed in [9] for some concrete examples. It consists in replacing the Hilbert space \mathcal{H} by the subspace \mathcal{H}_Λ ,

$$\mathcal{H}_\Lambda = \text{range } \chi \left(\frac{D}{\Lambda} \right) \quad (1.27)$$

and restricting both D and \mathcal{A} to this subspace, while maintaining the commutation rule (1.20) for the algebra \mathcal{A} . This procedure is superior to the familiar lattice approximation because it does respect the geometric symmetry group. The point is that finite dimensional *noncommutative* algebras have continuous Lie groups of automorphisms while the automorphism group of a commutative finite dimensional algebra is necessarily finite. The hypothesis which we shall test in this paper is that there exist an energy scale Λ in the range $10^{15} - 10^{19}$ GeV at which the bare action (1.2) becomes geometric, i.e. \sim

$$\text{Trace } \chi \left(\frac{D}{\Lambda} \right) + \langle \psi, D\psi \rangle. \quad (1.28)$$

2. The spectral action principle applied to the Einstein-Yang-Mills system.

To test the spectral action functional (1.28) we shall first consider the simplest noncommutative modification of a manifold M . Thus we replace the algebra

$C^\infty(M)$ of smooth functions on M by the tensor product $\mathcal{A} = C^\infty(M) \otimes M_N(\mathbb{C})$ where $M_N(\mathbb{C})$ is the algebra of $N \times N$ matrices. The spectral triple is obtained by tensoring the Dirac spectral triple for M by the spectral triple for $M_N(\mathbb{C})$ given by the left action of $M_N(\mathbb{C})$ on the Hilbert space of $N \times N$ matrices with Hilbert-Schmidt norm while the operator is 0. The real structure is given by the adjoint operation, $m \rightarrow m^*$ on matrices. Thus for the product geometry one has

$$\begin{aligned}\mathcal{H} &= L^2(M, S) \otimes M_N(\mathbb{C}) \\ J(\xi \otimes m) &= C\xi \otimes m^* \\ D &= \not{D}_M \otimes 1.\end{aligned}\tag{2.1}$$

We shall compare the spectral action functional (1.28) with the following

$$I = \frac{1}{2\kappa^2} \int R \sqrt{g} d^4x + I_{YM}\tag{2.2}$$

where $I_{YM} = \int (\mathcal{L}_G + \mathcal{L}_{Gf}) \sqrt{g} d^4x$ is the action for an $SU(N)$ Yang-Mills theory coupled to fermions in the adjoint representation.

Starting with (2.1), one first computes the internal fluctuations of the metric and finds that they are parametrized exactly by an $SU(N)$ Yang Mills field A . Note that the formula $D = D_0 + A + JAJ^*$ eliminates the $U(1)$ part of A even if one starts with an $U(N)$ gauge potential.

One also checks that the coupling of the Yang Mills field A with the fermions is equal to

$$\langle \psi, D\psi \rangle \quad \psi \in \mathcal{H}.\tag{2.3}$$

The operator $D = D_0 + A + JAJ^*$ is given by

$$D = e_a^\mu \gamma^a \left((\partial_\mu + \omega_\mu) \otimes 1_N + 1 \otimes \left(-\frac{i}{2} g_0 A_\mu^i T^i \right) \right)\tag{2.4}$$

where ω_μ is the spin-connection on M :

$$\omega_\mu = \frac{1}{4} \omega_\mu^{ab} \gamma_{ab}$$

and T^i are matrices in the adjoint representation of $SU(N)$ satisfying $\text{Tr}(T^i T^j) = 2\delta^{ij}$. (ω_μ^{ab} is related to the e_μ^a by the vanishing of the covariant derivative^(*),

$$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a - \omega_\mu^{ab} e_\nu^b - \Gamma_{\mu\nu}^\rho e_\rho^a = 0.\tag{2.5}$$

(*) We have limited our considerations to torsion free spaces. The more general case of torsion will be treated somewhere else.

As the Christoffel connection

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{\rho\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \quad (2.6)$$

is a given function of $g_{\mu\nu} = e_{\mu}^a e_{\nu}^a$, equation (2.5) could be solved to express ω_{μ}^{ab} as a function of e_{μ}^a .)

It is a simple exercise to compute the square of the Dirac operator given by (2.4) [10-11]. This can be cast into the elliptic operator form [12]:

$$P = D^2 = -(g^{\mu\nu} \partial_{\mu} \partial_{\nu} \cdot \mathbb{I} + \mathbb{A}^{\mu} \partial_{\mu} + \mathbb{B}) \quad (2.7)$$

where \mathbb{I} , \mathbb{A}^{μ} and \mathbb{B} are matrices of the same dimensions as D , and are given by:

$$\begin{aligned} \mathbb{A}^{\mu} &= (2\omega^{\mu} - \Gamma^{\mu}) \otimes 1_N - ig_0 1_4 \otimes A^{\mu i} T^i \\ \mathbb{B} &= (\partial^{\mu} \omega_{\mu} + \omega^{\mu} \omega_{\mu} - \Gamma^{\nu} \omega_{\nu} + R) \otimes 1_N - ig_0 \omega_{\mu} \otimes A^{\mu i} T^i. \end{aligned} \quad (2.8)$$

In deriving (2.8) we have used equation (2.5) as well as the following definitions and identities

$$\begin{aligned} [\partial_{\mu} + \omega_{\mu}, \partial_{\nu} + \omega_{\nu}] &\equiv \frac{1}{4} R_{\mu\nu}^{ab} (\omega(e)) \gamma_{ab} \\ e_{\rho}^a e_{\sigma}^b R_{\mu\nu}^{ab} (\omega(e)) &= R_{\mu\nu\rho\sigma}(g) \\ R_{\nu\rho\sigma}^{\mu} &= \partial_{\rho} \Gamma_{\nu\sigma}^{\mu} - \partial_{\sigma} \Gamma_{\nu\rho}^{\mu} + \Gamma_{\rho\kappa}^{\mu} \Gamma_{\nu\sigma}^{\kappa} - \Gamma_{\sigma\kappa}^{\mu} \Gamma_{\nu\rho}^{\kappa} \\ \Gamma^{\mu} &= g^{\nu\sigma} \Gamma_{\nu\sigma}^{\mu} \end{aligned} \quad (2.9)$$

we have also used the symmetries of the Riemann tensor to prove that

$$\gamma^{\mu\nu} R_{\mu\nu}^{ab} \gamma_{ab} = -2R. \quad (2.10)$$

We shall now compute the spectral action for this theory given by

$$\text{Tr} \chi \left(\frac{D^2}{m_0^2} \right) + (\psi, D\psi) \quad (2.11)$$

where the trace Tr is the usual trace of operators in the Hilbert space \mathcal{H} , and m_0 is a (mass) scale to be specified. The function χ is chosen to be positive and this has important consequences for the positivity of the gravity action.

Using identities [12]:

$$\text{Tr}(P^{-s}) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \text{Tr} e^{-tP} dt \quad \text{Re}(s) \geq 0 \quad (2.12)$$

and the heat kernel expansion for

$$\mathrm{Tr} e^{-tP} \simeq \sum_{n \geq 0} t^{\frac{n-m}{d}} \int_M a_n(x, P) dv(x) \quad (2.13)$$

where m is the dimension of the manifold in $C^\infty(M)$, d is the order of P (in our case $m = 4$, $d = 2$) and $dv(x) = \sqrt{g} d^m x$ where $g^{\mu\nu}$ is the metric on M appearing in equation (2.7).

If $s = 0, -1, \dots$ is a non-positive integer then $\mathrm{Tr}(P^{-s})$ is regular at this value of s and is given by

$$\mathrm{Tr}(P^{-s}) = \mathrm{Res} \Gamma(s) \big|_{s=\frac{m-n}{d}} a_n.$$

From this we deduce that

$$\mathrm{Tr} \chi(P) \simeq \sum_{n \geq 0} f_n a_n(P) \quad (2.14)$$

where the coefficients f_n are given by

$$\begin{aligned} f_0 &= \int_0^\infty \chi(u) u du, \quad f_2 = \int_0^\infty \chi(u) du, \\ f_{2(n+2)} &= (-1)^n \chi^{(n)}(0), \quad n \geq 0 \end{aligned} \quad (2.15)$$

and $a_n(P) = \int a_n(x, P) dv(x)$.

The Seeley-de Witt coefficients $a_n(P)$ vanish for odd values of n . The first three a_n 's for n even are [12]:

$$\begin{aligned} a_0(x, P) &= (4\pi)^{-m/2} \mathrm{Tr}(\mathbb{I}) \\ a_2(x, P) &= (4\pi)^{-m/2} \mathrm{Tr} \left(-\frac{R}{6} \mathbb{I} + \mathbb{E} \right) \\ a_4(x, P) &= (4\pi)^{-m/2} \frac{1}{360} \mathrm{Tr}((-12R;_\mu{}^\mu + 5R^2 - 2R_{\mu\nu} R^{\mu\nu} \\ &\quad + 2R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}) \mathbb{I} - 60R\mathbb{E} + 180\mathbb{E}^2 + 60\mathbb{E};_\mu{}^\mu \\ &\quad + 30\Omega_{\mu\nu} \Omega^{\mu\nu}) \end{aligned} \quad (2.16)$$

where \mathbb{E} and $\Omega_{\mu\nu}$ are defined by

$$\begin{aligned} \mathbb{E} &= \mathbb{B} - g^{\mu\nu} (\partial_\mu \omega'_\nu + \omega'_\mu \omega'_\nu - \Gamma_{\mu\nu}^\rho \omega'_\rho) \\ \Omega_{\mu\nu} &= \partial_\mu \omega'_\nu - \partial_\nu \omega'_\mu + [\omega'_\mu \omega'_\nu] \\ \omega'_\mu &= \frac{1}{2} g_{\mu\nu} (\mathbb{A}^\nu - \Gamma^\nu \cdot \mathbb{I}). \end{aligned} \quad (2.17)$$

The Ricci and scalar curvature are defined by

$$\begin{aligned} R_{\mu\rho} &= R_{\mu\nu}{}^{ab} e_b^\nu e_{a\rho} \\ R &= R_{\mu\nu}{}^{ab} e_a^\mu e_b^\nu. \end{aligned} \quad (2.18)$$

We now have all the necessary tools to evaluate explicitly the spectral action (2.11). Using equations (2.8) and (2.16) we find :

$$\begin{aligned} \mathbb{E} &= \frac{1}{4} R \otimes \mathbb{I}_4 \otimes \mathbb{I}_N + \frac{i}{4} \gamma^{\mu\nu} \otimes g F_{\mu\nu}^i T^i \\ \Omega_{\mu\nu} &= \frac{1}{4} R_{\mu\nu}{}^{ab} \gamma_{ab} \otimes 1_N - \frac{i}{2} \mathbb{I}_4 \otimes g F_{\mu\nu}^i T^i. \end{aligned} \quad (2.19)$$

From the knowledge that the invariants of the heat equation are polynomial functions of R , $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$, \mathbb{E} and $\Omega_{\mu\nu}$ and their covariant derivatives, it is then evident from equation (2.19) that the spectral action would not only be diffeomorphism invariant but also gauge invariant. The first three invariants are then^(*)

$$\begin{aligned} a_0(P) &= \frac{N}{4\pi^2} \int_M \sqrt{g} d^4x \\ a_2(P) &= \frac{N}{48\pi^2} \int_M \sqrt{g} R d^4x \\ a_4(P) &= \frac{1}{16\pi^2} \cdot \frac{N}{360} \int_M d^4x \sqrt{g} \left[(12R;_\mu{}^\mu + 5R^2 - 8R_{\mu\nu} R^{\mu\nu} \right. \\ &\quad \left. - 7R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}) + \frac{120}{N} g^2 F_{\mu\nu}^i F^{\mu\nu i} \right]. \end{aligned} \quad (2.20)$$

For the special case where the dimension of the manifold M is four, we have a relation between the Gauss-Bonnet topological invariant and the three possible curvature square terms:

$$R^* R^* = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \quad (2.21)$$

where $R^* R^* \equiv \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} R_{\mu\nu}^{\alpha\beta} R_{\rho\sigma}^{\gamma\delta}$. Moreover, we can change the expression for $a_4(P)$ in terms of $C_{\mu\nu\rho\sigma}$ instead of $R_{\mu\nu\rho\sigma}$ where

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - (g_{\mu[\rho} R_{\nu|\sigma]} - g_{\nu[\rho} R_{\mu|\sigma]}) + \frac{1}{6} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) R \quad (2.22)$$

(*) Note that according to our notations the scalar curvature R is negative for spheres.

is the Weyl tensor. Using the identity:

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + 2R_{\mu\nu} R^{\mu\nu} - \frac{1}{3}R^2 \quad (2.23)$$

we can recast $a_4(P)$ into the alternative form:

$$a_4(p) = \frac{N}{48\pi^2} \int d^4x \sqrt{g} \left[-\frac{3}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{1}{120} (11R^* R^* + 12R_{;\mu}{}^\mu) + \frac{g^2}{N} F_{\mu\nu}^i F^{\mu\nu i} \right] \quad (2.24)$$

and this is explicitly conformal invariant. The Euler characteristic χ_E (not to be confused with the function χ) is related to $R^* R^*$ by the relation

$$\chi_E = \frac{1}{32\pi^2} \int d^4x \sqrt{g} R^* R^* . \quad (2.25)$$

If we choose the function χ to be a cutoff function, i.e. $\chi(x) = 1$ for x near 0, then $\chi^{(n)}(0)$ is zero $\forall n > 0$ and this removes the non-renormalizable interactions. It is, also possible to introduce a mass scale m_0 and consider χ to be a function of the dimensionless variable $\chi\left(\frac{P}{m_0^2}\right)$. In this case terms coming from $a_n(P)$, $n > 4$ will be suppressed by powers of $\frac{1}{m_0^2}$:

$$I_b = \frac{N}{48\pi^2} \left[12m_0^4 f_0 \int d^4x \sqrt{g} + m_0^2 f_2 \int d^4x \sqrt{g} R + f_4 \int d^4x \sqrt{g} \left[-\frac{3}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{1}{10} R_{;\mu}{}^\mu + \frac{11}{20} R^* R^* + \frac{g^2}{N} F_{\mu\nu}^i F^{\mu\nu i} \right] + 0 \left(\frac{1}{m_0^2} \right) \right] . \quad (2.26)$$

We shall adopt Wilson's view point of the renormalization group approach to field theory [13] where the spectral action is taken to give the *bare* action with bare quantities m_0 and g_0 and with a cutoff scale Λ where the theory is assumed to take a geometrical form. Introducing the cutoff scale Λ will regularize the theory. The perturbative expansion is then reexpressed in terms of *renormalized* physical quantities. The fields also receive wave function renormalization. Normalizing the Einstein and Yang-Mills terms in the bare action we then have:

$$\frac{N m_0^2 f^2}{24\pi^2} = \frac{1}{\kappa_0^2} \equiv \frac{1}{8\pi G_0} \quad \frac{f_4 g_0^2}{12\pi^2} = 1 \quad (2.27)$$

and (2.26) becomes:

$$I_b = \int d^4x \sqrt{g} \left[\frac{1}{2\kappa_0^2} R + e_0 + a_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + c_0 R^* R^* + d_0 R_{;\mu}{}^\mu + \frac{1}{4} F_{\mu\nu}^i F^{\mu\nu i} \right] \quad (2.28)$$

where

$$\begin{aligned} a_0 &= \frac{-3N}{80} \frac{1}{g_0^2} \\ c_0 &= -\frac{2}{3} a_0 \\ d_0 &= -\frac{11}{3} a_0 \\ e_0 &= \frac{N m_0^4}{4\pi^2} f_0. \end{aligned} \quad (2.29)$$

The renormalized action receives counterterms of the same form as the bare action but with physical parameters k, a, c, d , and the addition of one new term [14]

$$\int d^4x \sqrt{g} (b R^2). \quad (2.30)$$

This adds one further boundary condition for the equations (2.29):

$$b_0 = 0.$$

The renormalized fermionic action $(\psi, D\psi)$ keeps the same form as the bare fermionic action.

The renormalization group equations will yield relations between the bare quantities and the physical quantities with the addition of the cutoff scale Λ . Conditions on the bare quantities would translate into conditions on the physical quantities. In the present example only the gauge coupling $g(\Lambda)$ and Newton's constant will have measurable effects. The dependence of κ_0 on κ and the other physical quantities is such that $\kappa_0^{-2} - \kappa^{-2}$ contains terms proportional to the cutoff scale. As κ^2 must be identified with $8\pi G$ at low energy it is clear that both κ_0^{-1} and Λ could be as high as the Planck scale^(*)

The renormalization group equations of this system were studied by Fradkin and Tseytlin [15] and is known to be renormalizable, but non-unitary [14] due to the

(*) We like to thank A. Tseytlin for correspondence on this point.

presence of spin-two ghost (tachyon) pole near the Planck mass. We shall not worry about non-unitarity (see, however, reference 16), because in our view at the Planck energy the manifold structure of space-time will break down and one must have a completely finite theory where only the part of the Hilbert space given by $\chi(D^2)\mathcal{H}$ enters. The algebra \mathcal{A} becomes finite dimensional in such a way that all symmetries of the continuum (in some approximation) would be admitted.

In the limit of flat space-time we have $g_{\mu\nu} = \delta_{\mu\nu}$ and the action (2.11) becomes (adopting the normalizations (2.29)):

$$\frac{1}{4} F_{\mu\nu}^i F^{\mu\nu i} + (\psi, D\psi) \quad (2.31)$$

where we have dropped the constant term. This action is known to have $N = 1$ global supersymmetry. In reality we can also obtain the $N = 2$ and $N = 4$ super Yang-Mills actions by taking the appropriate Dirac operators in six and ten dimensions respectively [17].

3. Spectral action for the standard model.

Having illustrated the computation of our spectral action for the Einstein-Yang-Mills system we now address the realistic case of obtaining action (1.2) for the Einstein-Standard model system.

We first briefly summarize the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ associated with the spectrum of the standard model. A complete treatment can be found in [4,6].

The geometry is that of a 4-dimensional smooth Riemannian manifold with a fixed spin structure times a discrete geometry. The product geometry is given by the rules

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \quad D = D_1 \otimes 1 + \gamma_5 \otimes D_2 \quad (3.1)$$

where $\mathcal{A}_1 = C^\infty(M)$, $D_1 = \not{D}_M$ the Dirac operator on M , $\mathcal{H}_1 = L^2(M, S)$ and the discrete geometry $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ will now be described. The algebra \mathcal{A}_2 is the direct sum of the real involutive algebras \mathbb{C} of complex numbers, \mathbb{H} of quaternions, and $M_3(\mathbb{C})$ of 3×3 matrices. \mathcal{H}_2 is the Hilbert space with basis the elementary fermions, namely the quarks Q , leptons L and their charge conjugates, where

$$Q = \begin{pmatrix} u_L \\ d_L \\ d_R \\ u_R \end{pmatrix}, \quad L = \begin{pmatrix} \nu_L \\ e_L \\ e_R \end{pmatrix} \quad (3.2)$$

and we have omitted family indices for Q and L and colour index for Q . The antilinear isometry $J = J_2$ in \mathcal{H}_2 exchanges a fermions with its conjugate.

The action of an element $a = (\lambda, q, m) \in \mathcal{A}_2$ in \mathcal{H}_2 is given by:

$$a Q = \begin{pmatrix} q \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\ \bar{\lambda} d_R \\ \lambda u_R \end{pmatrix} \quad (3.3)$$

where $q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ is a quaternion. A similar formula holds for leptons. The action on conjugate particles is:

$$\begin{aligned} a \bar{L} &= \lambda \bar{L} \\ a \bar{Q} &= m \bar{Q}. \end{aligned} \quad (3.4)$$

For the operator D_2 we take $D_2 = \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix}$ where Y is a Yukawa coupling matrix of the form

$$Y = Y_q \otimes 1_3 \oplus Y_\ell \quad (3.5)$$

with

$$\begin{aligned} Y_q &= \begin{pmatrix} 0_2 & k_0^d \otimes H_0 & k_0^u \otimes \tilde{H}_0 \\ (k_0^d)^* \otimes H_0^* & (k_0^u)^* \otimes \tilde{H}_0^* & 0_2 \end{pmatrix} \\ Y_\ell &= \begin{pmatrix} 0_2 & k_0^e \otimes H_0 \\ k_0^{e*} \otimes H_0^* & 0 \end{pmatrix}. \end{aligned} \quad (3.6)$$

The matrices k^d , k^u and k^e are 3×3 family mixing matrices and

$$H_0 = \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{H}_0 = i\sigma_2 H_0.$$

The parameter μ has the dimension of mass.

The choice of the Dirac operator and the action of \mathcal{A}_2 in \mathcal{H}_2 comes from the restrictions that these must satisfy:

$$\begin{aligned} J^2 &= 1, \quad [J, D_2] = 0, \quad [a, Jb^* J^{-1}] = 0 \\ [[D, a], Jb^* J^{-1}] &= 0 \quad \forall a, b. \end{aligned} \quad (3.7)$$

The next step is to compute the inner fluctuations of the metric and thus the operators of the form: $A = \Sigma a_i [D, b_i]$. This with the self-adjointness condition

$A = A^*$ gives a $U(1)$, $SU(2)$ and $U(3)$ gauge fields as well as a Higgs field. The computation of $A + JAJ^{-1}$ removes a $U(1)$ part from the above gauge fields (such that the full matrix is traceless) (for derivation see [4]). The Dirac operator D_q that takes the inner fluctuations into account is given by the 36×36 matrix (acting on the 36 quarks) (tensored with Clifford algebras)

$$D_q = \begin{bmatrix} \gamma^\mu \otimes (D_\mu \otimes 1_2 - \frac{i}{2}g_{02}A_\mu^\alpha \sigma^\alpha - \frac{i}{6}g_{01}B_\mu \otimes 1_2) \otimes 1_3, & \gamma_5 \otimes k_0^d \otimes H, & \gamma_5 \otimes k_0^u \otimes \tilde{H} \\ \gamma_5 \otimes k_0^{d*} \otimes H^*, & \gamma^\mu \otimes (D_\mu + \frac{i}{3}g_{01}B_\mu) \otimes 1_3, & 0 \\ \gamma_5 k_0^{u*} \tilde{H}^*, & 0, & \gamma^\mu \otimes (D_\mu - \frac{2i}{3}g_{01}B_\mu) \otimes 1_3 \end{bmatrix} \otimes 1_3 \\ + \gamma^\mu \otimes 1_4 \otimes 1_3 \otimes \left(-\frac{i}{2}g_{03}V_\mu^i \lambda^i \right) \quad (3.8)$$

where σ^α are Pauli matrices and λ^i are Gell-mann matrices satisfying

$$\text{Tr}(\lambda^i \lambda^j) = 2\delta^{ij}. \quad (3.9)$$

The vector fields B_μ , A_μ^α and V_μ^i are the $U(1)$, $SU(2)_w$ and $SU(3)_c$ gauge fields with gauge couplings g_{01} , g_{02} and g_{03} . The differential operator D_μ is given by

$$D_\mu = \partial_\mu + \omega_\mu \quad (3.10)$$

and $\gamma^\mu = e_a^\mu \gamma^a$. The scalar field H is the Higgs doublet, and \tilde{H} is the $SU(2)$ conjugate of H :

$$\tilde{H} = (i\sigma^2 H). \quad (3.11)$$

We note that although H_0 was introduced in the definition of D_2 it is absorbed in the field H .

It is a simple exercise to see that the action for the fermionic quark sector is given by

$$(Q, D_q Q). \quad (3.12)$$

The Dirac operator acting on the leptons, taking inner fluctuations into account is given by the 9×9 matrix (tensored with Clifford algebra matrices):

$$D_\ell = \begin{bmatrix} \gamma^\mu \otimes (D_\mu - \frac{i}{2}g_{02}A_\mu^\alpha \sigma^\alpha + \frac{i}{2}g_{01}B_\mu \otimes 1_2) \otimes 1_3 & \gamma_5 \otimes k_0^e \otimes H \\ \gamma_5 \otimes k_0^{*e} \otimes H^* & \gamma^\mu \otimes (D_\mu + ig_{01}B_\mu) \otimes 1_3 \end{bmatrix}. \quad (3.13)$$

Again the leptonic action have the simple form

$$(L, D_\ell L). \quad (3.14)$$

According to our universal formula (1.28) the spectral action for the standard model is given by:

$$\text{Tr}[\chi(D^2/m_0^2)] + (\psi, D\psi) \quad (3.15)$$

where $(\psi, D\psi)$ will include the quark sector (3.12) and the leptonic sector (3.14). Calculating the bosonic part of the above action follows the same lines as in the previous section. The steps that lead to the result are given in the appendix.

The bosonic action is

$$\begin{aligned} I = & \frac{9m_0^4}{\pi^2} \frac{5}{4} f_0 \int d^4x \sqrt{g} \\ & + \frac{3m_0^2}{4\pi^2} f_2 \int d^4x \sqrt{g} \left[\frac{5}{4} R - 2y^2 H^* H \right] \\ & + \frac{f_4}{4\pi^2} \int d^4x \sqrt{g} \left[\frac{1}{40} \frac{5}{4} (12R;_{\mu}{}^{\mu} + 11R^* R^* - 18C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}) \right. \\ & + 3y^2 \left(D_{\mu} H^* D^{\mu} H - \frac{1}{6} R H^* H \right) \\ & + g_{03}^2 G_{\mu\nu}^i G^{\mu\nu i} + g_{02}^2 F_{\mu\nu}^{\alpha} F^{\mu\nu\alpha} \\ & + \frac{5}{3} g_{01}^2 B_{\mu\nu} B^{\mu\nu} \\ & \left. + 3z^2 (H^* H)^2 - y^2 (H^* H);_{\mu}{}^{\mu} \right] + 0 \left(\frac{1}{m_0^2} \right) \end{aligned} \quad (3.16)$$

where we have denoted

$$\begin{aligned} y^2 &= \text{Tr} \left(|k_0^d|^2 + |k_0^u|^2 + \frac{1}{3} |k_0^e|^2 \right) \\ z^2 &= \text{Tr} \left((|k_0^d|^2 + |k_0^u|^2)^2 + \frac{1}{3} |k_0^e|^4 \right) \\ D_{\mu} H &= \partial_{\mu} H - \frac{i}{2} g_{02} A_{\mu}^{\alpha} \sigma^{\alpha} H - \frac{i}{2} g_{01} B_{\mu} H . \end{aligned} \quad (3.17)$$

Normalizing the Einstein and Yang-Mills terms gives:

$$\begin{aligned} \frac{15m_0^2 f_2}{4\pi^2} &= \frac{1}{\kappa_0^2} \\ \frac{g_{03}^2 f_4}{\pi^2} &= 1 \\ g_{03}^2 &= g_{02}^2 = \frac{5}{3} g_{01}^2 . \end{aligned} \quad (3.18)$$

Relations (3.18) among the gauge coupling constants coincide with those coming from $SU(5)$ unification.

To normalize the Higgs fields kinetic energy we have to rescale H by:

$$H \rightarrow \frac{2}{3} \frac{g_{03}}{y} H. \quad (3.19)$$

This transforms the bosonic action (3.16) to the form:

$$\begin{aligned} I_b = \int d^4x \sqrt{g} \left[\frac{1}{2\kappa_0^2} R - \mu_0^2 (H^* H) + a_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right. \\ + b_0 R^2 + c_0 {}^* R^* R + d_0 R_{;\mu}{}^\mu \\ + e_0 + \frac{1}{4} G_{\mu\nu}^i G^{\mu\nu i} + \frac{1}{4} F_{\mu\nu}^\alpha F^{\mu\nu\alpha} \\ \left. + \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + |D_\mu H|^2 - \xi_0 R |H|^2 + \lambda_0 (H^* H)^2 \right] \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} \mu_0^2 &= \frac{4}{3\kappa_0^2} \\ a_0 &= -\frac{9}{8g_{03}^2} \\ b_0 &= 0 \\ c_0 &= -\frac{11}{18} a_0 \\ d_0 &= -\frac{2}{3} a_0 \\ e_0 &= \frac{45}{4\pi^2} f_0 m_0^4 \\ \lambda_0 &= \frac{4}{3} g_{03}^2 \frac{z^2}{y^4} \\ \xi_0 &= \frac{1}{6}. \end{aligned} \quad (3.21)$$

As explained in the last section this action has to be taken as the *bare action* at some cutoff scale Λ . The renormalized action will have the same form as (3.20) but with the bare quantities κ_0 , μ_0 , λ_0 , a_0 to e_0 and g_{01} , g_{02} , g_{03} replaced with physical quantities.

Relations between the bare gauge coupling constants as well as equations (3.19) have to be imposed as boundary conditions on the renormalization group equations

[13]. The bare mass of the Higgs field is related to the bare value of Newton's constant, and both have quadratic divergences in the limit of infinite cutoff Λ . The relation between m_0^2 and the physical quantities is:

$$m_0^2 = m^2 \left(1 + \frac{\left(\frac{\Lambda^2}{m^2} - 1 \right)}{32\pi^2} \left(\frac{9}{4} g_2^2 + \frac{3}{4} g_1^2 + 6\lambda - 6k_t^2 \right) \right) + 0 \left(\ln \frac{\Lambda^2}{m^2} \right) + \dots \quad (3.22)$$

For $m^2(\Lambda)$ to be small at low-energies m_0^2 should be tuned to be proportional to the cutoff scale according to equation (3.22).

Similarly the bare cosmological constant is related to the physical one (which must be tuned to zero at low energies):

$$e_0 = e + \frac{\Lambda^4}{32\pi^2} (62) + \dots \quad (3.23)$$

where 62 is the difference between the fermionic degrees of freedom (90) and the bosonic ones (28).

There is also a relation between the bare scale κ_0^{-2} and the physical one κ^{-2} which is similar to equation (3.20) (but with all one-loop contributions coming with the same sign) which shows that $\kappa_0^{-1} \sim m_0$ and Λ are of the same order as the Planck mass.

The renormalization group equations for the gauge coupling constants are:

$$\begin{aligned} \frac{dg_1}{dt} &= \frac{1}{16\pi^2} \left(\frac{41}{6} \right) g_1^3 \\ \frac{dg_2}{dt} &= \frac{1}{16\pi^2} \left(-\frac{19}{6} \right) g_2^3 \\ \frac{dg_3}{dt} &= \frac{1}{16\pi^2} (-7) g_3^3 \end{aligned} \quad (3.24)$$

where $t = \ln \mu$, μ being the running scale.

Solutions to equations (3.22) are known from the $SU(5)$ case and are given by [19]

$$\begin{aligned} \alpha_1^{-1}(M_Z) &= \alpha_1^{-1}(\Lambda) + \frac{41}{12\pi} \ln \frac{\Lambda}{M_Z} \\ \alpha_2^{-1}(M_Z) &= \alpha_2^{-1}(\Lambda) - \frac{19}{12\pi} \ln \frac{\Lambda}{M_Z} \\ \alpha_3^{-1}(M_Z) &= \alpha_3^{-1}(\Lambda) - \frac{42}{12\pi} \ln \frac{\Lambda}{M_Z} \end{aligned} \quad (3.25)$$

where $\alpha_i = \frac{g_i^2}{4\pi}$, $i = 1, 2, 3$ and M_Z is the mass of the Z vectors. At the scale Λ we have to impose the boundary conditions (3.18):

$$\alpha_3(\Lambda) = \alpha_2(\Lambda) = \frac{5}{3} \alpha_1(\Lambda). \quad (3.26)$$

Using equations (3.27) and (3.28) one easily find:

$$\begin{aligned} \sin^2 \theta_w &= \frac{3}{8} \left[1 - \frac{109}{18\pi} \alpha_{em} \ln \frac{\Lambda}{M_Z} \right] \\ \ln \frac{\Lambda}{M_Z} &= \frac{2\pi}{67} (3\alpha_{em}^{-1}(M_Z) - 8\alpha_3^{-1}(M_Z)). \end{aligned} \quad (3.27)$$

The present experimental values for $\alpha_{em}^{-1}(M_Z)$ and $\alpha_3(M_Z)$ are

$$\begin{aligned} \alpha_{em}^{-1}(M_Z) &= 128.09 \\ 0.110 &\leq \alpha_3(M_Z) \leq 0.123. \end{aligned} \quad (3.28)$$

These values lead to

$$\begin{aligned} 9.14 \times 10^{14} &\leq \Lambda \leq 4.44 \times 10^{14} (\text{Gev}) \\ 0.206 &\leq \sin^2 \theta_w \leq 0.210. \end{aligned} \quad (3.29)$$

Therefore the bare action we obtained and associated with the spectrum of the standard model is consistent with experimental data provided the cutoff scale is taken to be $\Lambda \sim 10^{15}$ Gev. There is, however, a slight disagreement (10%) between the predicted value of $\sin^2 \theta_w$ and the experimental value of 0.2325 known to a very high precision. It is a remarkable fact that starting from the spectrum of the standard model at low-energies, and assuming that this spectrum does not change, one can get the geometrical spectral action which holds at very high-energies and consistent within ten percent with experimental data. This can be taken that at higher energies the noncommutative nature of space-time reveals itself and shows that the effective theory at the scale Λ have a higher symmetry. The other disagreement is that the gravity sector requires the cutoff scale to be of the same order as the Planck scale while the condition on gauge coupling constants give $\Lambda \sim 10^{15}$ Gev. The gravitational coupling G runs with Λ due to the matter interactions. This dictates that it must be of the order Λ^{-2} and gives a large value for Newton's constant. These results must be taken as an indication that the spectrum of the standard model has to be altered as we climb up in energy.

The change may happen at low energies (just as in supersymmetry which also pushes the cutoff scale to 10^{16} GeV) or at some intermediate scale. Incidentally the problem that Newton's constant is coming out to be too large is also present in string theory where also a unification of gauge couplings and Newton's constant occurs [20]. Ultimately one would hope that modification of the spectrum will increase the cutoff scale nearer to the Planck mass as dictated by gravity.

There is one further relation in our theory between the $\lambda(H^*H)^2$ coupling and the gauge couplings to be imposed at the scale Λ [21]:

$$\lambda_0 = \frac{4}{3} g_{03}^2 \frac{z^2}{y^4} . \quad (3.30)$$

This relation could be simplified if we assume that the top quark Yukawa coupling is much larger than all the other Yukawa couplings. In this case equation (3.30) simplifies to

$$\lambda(\Lambda) = \frac{16\pi}{3} \alpha_3(\Lambda) . \quad (3.31)$$

Therefore the value of λ at the unification scale is $\lambda_0 \simeq 0.402$ showing that one does not go outside the perturbation domain. In reality, equation (3.31) could be used, together with the RG equations for λ and k_t to determine the Higgs mass at the low-energy scale M_Z [22]:

$$\begin{aligned} \frac{d\lambda}{dt} &= 4\lambda\gamma + \frac{1}{8\pi^2}(12\lambda^2 + B) \\ \frac{dk_t}{dt} &= \frac{1}{32\pi^2} \left[9k_t^3 - \left(16g_3^2 + \frac{9}{2}g_2^2 + \frac{17}{6}g_1^2 \right) k_t \right] \end{aligned} \quad (3.32)$$

where

$$\begin{aligned} \gamma &= \frac{1}{64\pi^2}(12k_t^2 - 9g_t^2 - 3g_1^2) \\ B &= \frac{3}{84\pi^2} \left(\frac{1}{16}(3g_2^4 + 2g_1^2g_2^2 - g_1^4) - k_t^4 \right) . \end{aligned} \quad (3.33)$$

These equations have to be integrated numerically [21]. One can get a rough estimate on the Higgs mass from the triviality bound^(*) on the λ couplings. For $\Lambda \simeq 10^{15}$ GeV as given in equation (3.29) the limits are

$$160 < m_H < 200 \text{ GeV} . \quad (3.34)$$

(*) We like to thank M. Lindner for explanations on this point.

This together with the boundary condition (3.31) gives a mass of the Higgs near the lower bound of 160 GeV. The exact answer can be only determined by numerical integration, but this of course cannot be completely trusted as the predicted value for $\sin^2 \theta_w$ is off by ten percent. It can, however, be taken as an approximate answer and in this respect one can say that the Higgs mass lies in the interval 160 – 180 GeV. We expect this prediction to be correct to the same precision as that of $\sin^2 \theta_w$ in (3.29).

In reality we can perform the same analysis for the gravitational sector to determine the dependence of κ_0 , a_0 , b_0 , c_0 , d_0 and e_0 on the physical quantities and the effect of the boundary conditions (3.19) on them. This, however, will not have measurable consequences and will not be pursued here.

4. Conclusions.

The basic symmetry for a noncommutative space $(\mathcal{A}, \mathcal{H}, D)$ is $\text{Aut}(\mathcal{A})$. This symmetry includes diffeomorphisms and internal symmetry transformations. The bosonic action is a spectral function of the Dirac operator while the fermionic action takes the simple linear form $(\psi, D\psi)$ where ψ are spinors defined on the Hilbert space. Applying this principle to the simple case where the algebra is $C^\infty(M) \otimes M_n(\mathbb{C})$ with a Hilbert space of fermions in the adjoint representation one finds that the bosonic action contains the Yang-Mills, Einstein and Weyl actions. This action is to be interpreted as the bare Wilsonian action at some cutoff scale Λ . The same principle when applied to the less trivial noncommutative geometry of the standard model gives the standard model action coupled to Einstein and Weyl gravity plus higher order non-renormalizable interactions suppressed by powers of the inverse of the mass scale in the theory. One also gets a mass term for the Higgs field. This bare mass is of the same order as the cutoff scale and this is related to the fact that there are quadratic divergences associated with the Higgs mass in the standard model. There are some relations between the bare quantities. The renormalized action will have the same form as the bare action but with physical quantities replacing the bare ones (except for an R^2 term which is absent in the bare action due to the scale invariance of the a_4 term associated with the square of the Dirac operator). The relations among the bare quantities must be taken as boundary conditions on the renormalization group equations governing the scale dependence of the physical quantities.

In particular there are relations among the gauge couplings coinciding with

those of $SU(5)$ (or any gauge group containing $SU(5)$) and also between the Higgs couplings to be imposed at some scale. These relations give a unification scale (or cutoff scale) of order $\sim 10^{15}$ Gev and a value for $\sin^2 \theta_w \sim 0.21$ which is off by ten percent from the true value. We also have a prediction of the Higgs mass in the interval $160 - 180$ Gev. This can be taken as an indication that the noncommutative structure of space-time reveals itself at such high scale where the effective action has a geometrical interpretation.

The slight disagreement with experiment indicates that the spectrum of the standard model could not be extrapolated to very high energies without adding new particles necessary to change the RG equations of the gauge couplings. One possibility could be supersymmetry, but there could be also less drastic solutions. It might be tempting by changing the spectrum to push the unification scale up nearer to the Planck scale a situation which is also present in string theory.

In summary, we have succeeded in finding a universal action formula that unified the standard model with the Einstein action. This necessarily involved an extrapolation from the low-energy sector to 10^{15} Gev, assuming no new physics arise. Our slight disagreement for the prediction of $\sin^2 \theta_w$ and for the low value of the unification scale seems to imply that the spectrum of the standard model must be modified either at low-energy or at an intermediate scale. There is also the possibility that by formulating the theory at very high energies, the concept of space-time as a manifold breaks down and the noncommutativity of the algebra must be extended to include the manifold part. One expects that the algebra A becomes a finite dimensional algebra. Finally, we hope that our universal action formula should be applicable to many situations of which the most important could be superconformal field theory. Work along these ideas is now in progress.

Acknowledgments. A.H.C. would like to thank Jürg Fröhlich for very useful discussions and I.H.E.S. for hospitality where part of this work was done.

Appendix.

To derive a general formula for $\text{Tr } \chi(D^2/\Lambda^2)$ we must evaluate the heat kernel invariants $a_n(x, P)$ for a Dirac operator of the form

$$D = \begin{pmatrix} \gamma^\mu (D_\mu \cdot \mathbb{I}_N + A_\mu) & \gamma_5 S \\ \gamma_5 S & \gamma^\mu (D_\mu \otimes \mathbb{I}_N + A_\mu) \end{pmatrix}. \quad (A.1)$$

Evaluating D^2 we find that

$$\mathbb{A}^\mu = ((2\omega^\mu - \Gamma^\mu) \otimes 1_N + 2A^\mu) \otimes 1_2 \quad (A.2)$$

$$\begin{aligned} \mathbb{B} = & (\partial^\mu \omega_\mu + \omega^\mu \omega_\mu - \Gamma^\mu \omega_\mu + R) \otimes 1_N + 2A^\mu \cdot \omega_\mu \\ & + \left(\partial^\mu + \omega^\mu - \Gamma^\mu \right) A_\mu - \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} + A^\mu A_\mu - S^2 \otimes 1_2 \\ & - \gamma^\mu \gamma_5 (D_\mu S + [A_\mu, S]) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (A.3)$$

From this we can construct \mathbb{E} and $\Omega_{\mu\nu}$:

$$\begin{aligned} \mathbb{E} = & \left(\frac{1}{4} R \otimes 1_N - \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} - S^2 \right) \otimes 1_2 \\ & - \gamma^\mu \gamma_5 (D_\mu S + [A_\mu, S]) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \quad (A.4)$$

$$\Omega_{\mu\nu} = \left(\frac{1}{4} R_{\mu\nu}{}^{ab} \gamma_{ab} \otimes 1_N + \gamma^{\mu\nu} F_{\mu\nu} \right) \otimes 1_2. \quad (A.5)$$

From this we deduce that

$$\begin{aligned} a_0(x, P) &= \frac{\Lambda^4}{4\pi^2} \int \sqrt{g} d^4x \text{Tr}(1) \\ a_2(x, P) &= \frac{\Lambda^2}{4\pi^2} \int \sqrt{g} d^4x \left[\frac{R}{12} \text{Tr}(1) - 2 \text{Tr}(S^2) \right] \\ a_4(x, P) &= \frac{1}{4\pi^2} \int \sqrt{g} d^4x \left[\frac{\text{Tr}(1)}{360} \left(3R_{;\mu}{}^\mu - \frac{9}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{11}{4} R^* R^* \right) \right. \\ &\quad + \text{Tr} \left((D_\mu S + [A_\mu, S])^2 - \frac{R}{6} S^2 \right) \\ &\quad \left. - \frac{1}{6} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr} S^4 - \frac{1}{3} \text{Tr}(S^2)_{;\mu}{}^\mu \right]. \end{aligned} \quad (A.6)$$

Applying these formulas to the Dirac operator of the quark sector we can obtain the same answer as from an explicit calculation by replacing in the previous formulas:

$$\begin{aligned}
\text{Tr}(1) &\rightarrow 36 \\
\text{Tr } S^2 &\rightarrow 3 \text{Tr}(|k_0^d|^2 + |k_0^u|^2) H^* H \\
\text{Tr } S^4 &\rightarrow 3 \text{Tr}(|k_0^d|^2 + |k_0^u|^2)^2 (H^* H)^2 \\
A_\mu &\rightarrow \begin{pmatrix} \left(-\frac{i}{2} g_{02} A_\mu^\alpha \sigma^\alpha - \frac{i}{6} g_{01} B_\mu \cdot \mathbb{I}_2\right) & \frac{1}{3} g_{01} B_\mu \\ \frac{1}{3} g_{01} B_\mu & -\frac{2i}{3} g_{01} B_\mu \end{pmatrix} \otimes 1_3 \otimes 1_3 \\
&\quad + 1_4 \otimes 1_3 \otimes \left(-\frac{i}{2} g_{03} V_\mu^i \lambda^i\right). \tag{A.7}
\end{aligned}$$

Then

$$-\frac{1}{6} \text{Tr } F_{\mu\nu} F^{\mu\nu} = \frac{3}{4} g_{02}^2 F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + \frac{11}{2} g_{01}^2 B_{\mu\nu} B^{\mu\nu} + g_{03}^2 G_{\mu\nu}^i G^{\mu\nu i}. \tag{A.8}$$

In the leptonic sector, we make the replacements:

$$\begin{aligned}
\text{Tr}(1) &\rightarrow 9 \\
\text{Tr } S^2 &\rightarrow \text{Tr } |k_0^e|^2 H^* H \\
\text{Tr } S^4 &\rightarrow \text{Tr } |k_0^e|^4 (H^* H)^2 \\
-\frac{1}{6} \text{Tr } F_{\mu\nu} F^{\mu\nu} &\rightarrow \frac{1}{3} \left(\frac{3}{4} g_{02}^2 F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + \frac{11}{2} g_{01}^2 B_{\mu\nu} B^{\mu\nu} + g_{03}^2 G_{\mu\nu}^i G^{\mu\nu i} \right) \tag{A.9}
\end{aligned}$$

References.

- [1] A. Connes, *Publ. Math. IHES* **62**, 44 (1983); Noncommutative Geometry (Academic Press, New York 1994).
- [2] A. Connes and J. Lott, *Nucl. Phys. Proc. Supp.* **B18**, 295 (1990); proceedings of 1991 Cargèse Summer Conference, edited by J. Fröhlich et al (Plenum, New York 1992).
- [3] D. Kastler, *Rev. Math. Phys.* **5**, 477 (1993).
- [4] A. Connes, “Gravity Coupled with Matter and the Foundation of Noncommutative Geometry”, hep-th/9603053.
- [5] A.H. Chamseddine, G. Felder and J. Fröhlich, *Comm. Math. Phys.* **155**, 109 (1993); A.H. Chamseddine, J. Fröhlich and O. Grandjean, *J. Math. Phys.* **36**, 6255 (1995).
- [6] A. Connes, *J. Math. Phys.* **36**, 6194 (1995).
- [7] B. De Witt, *Dynamical Theory of Groups and Fields* (New York, Gordon and Breach 1965).
- [8] S. Adler in *The high energy limit*, Erice lectures edited by A. Zichichi (Plenum, New York 1983).
- [9] H. Grosse, C. Klimcik and P. Presnajder, “On Finite 4D Quantum Field Theory in Noncommutative Geometry”, hep-th/9602115.
- [10] D. Kastler, *Comm. Math. Phys.* **166**, 633 (1995).
- [11] W. Kalau and M. Walze, *J. Geom. Phys.* **16**, 327 (1995).
- [12] P. Gilkey, Invariance theory, the heat equation and the Atiyah-Singer index theorem, (Publish or Perish, Dilmington, 1984).
- [13] K.G. Wilson, *Rev. Mod. Phys.* **47** (1975), 773; For an exposition very close to the steps taken here see C. Itzykson and J. -M. Drouffe, *Field theory*, Chapter five, Cambridge University Press, 1989.
- [14] K.S. Stelle, *Phys. Rev.* **D16**, 953 (1977).
- [15] E. Fradkin and A. Tseytlin, *Nucl. Phys.*
- [16] E. Tomboulis, *Phys. Lett.* **70B**, 361 (1977).
- [17] A.H. Chamseddine, *Phys. Lett.* **B332**, 349 (1994).
- [18] For a discussion of quadratic divergences and the hierarchy problem in the standard model see e.g. J. Ellis, “Supersymmetry and Grand Unification”, hep-ph/9512335.
- [19] For a review see G. Ross, Grand unified theories, *Frontiers in Physics Series*, Vol.60 (Benjamin, New York 1985).

- [20] E. Witten, “Strong Coupling Expansion of Calabi-Yau Compactification”, hep-th/9602070.
- [21] M. Bég, C. Panagiotakopoulos and A. Sirlin, *Phys. Rev. Lett.* **52**, 883 (1984);
M. Lindner, *Z. Phys.* **C31**, 295 (1986).
- [22] For an extensive review see M. Sher, *Phys. Rep.* **179**, 273 (1989).